

Oscillations toward the singularity of LRS Bianchi type IX cosmological models with Vlasov matter

Simone Calogero*

Departamento de Matemática Aplicada
Facultad de Ciencias, Universidad de Granada
18071 Granada, Spain

J. Mark Heinzle†

Gravitational Physics
Faculty of Physics, University of Vienna
1090 Vienna, Austria

Abstract

We analyze the dynamics of a class of cosmological solutions of the Einstein-Vlasov equations. These equations describe an ensemble of collisionless particles (which represent galaxies or clusters of galaxies) that interact gravitatively through Einstein's equations of general relativity. The cosmological models we consider are spatially homogeneous, of Bianchi type IX, and locally rotationally symmetric (LRS). We prove that generic solutions exhibit an oscillatory approach toward the singularities (the 'big bang' in the past and the 'big crunch' in the future); this is in contrast to the behavior of Einstein-vacuum or Einstein-Euler solutions. To establish this result we make use of dynamical systems theory; we introduce dimensionless dynamical variables that are defined on a compact state space; in this formulation the oscillatory behavior of generic solutions is represented by an approach to heteroclinic cycles.

1 Introduction

A large number of recent rigorous results on the dynamics of cosmological solutions (i.e., models for the 'universe') of the Einstein field equations is due to the successful application of the theory of dynamical systems to general relativity. The Einstein equations form a highly non-linear system of partial differential equations that describe the evolution of the metric of the space-time, the latter being represented by a four-dimensional Lorentzian manifold. In the case of spatially homogeneous solutions, which are of interest in cosmology, the Einstein equations reduce to a system of non-linear *ordinary* differential equations, which can be analyzed using the powerful methods of dynamical systems theory. This is typically achieved by going over from the standard metric variables of the Einstein equations to dimensionless variables via conformal rescalings and the introduction of normalizations to regularize the equations and to obtain an autonomous finite-dimensional dynamical system over a compact state space. We refer to [21] for an overview on the theory of dynamical systems in cosmology.

*E-Mail: calogero@ugr.es

†E-Mail: Mark.Heinzle@univie.ac.at

This paper concerns cosmological solutions of the Einstein equations coupled to collisionless matter (Vlasov matter). Solutions of the Einstein-Vlasov equations represent ensembles of massive particles (like stars in a galaxy) that interact through the gravitational field they create collectively. In cosmological applications, the particles are thought to represent galaxies (or galaxy clusters) in the universe.

According to the standard theory for the evolution of the universe, massless particles (photons) account for most of the energy density in the universe during the time of ‘radiation dominance’, which begins at about 1 second after the big bang and ends at the time of decoupling between radiation and matter about 10^5 years after the big bang. From this time onward, the universe is ‘matter dominated’ and galaxies begin to form about 10^6 years after the big bang. It is immediate that the collisionless matter model that describes ensembles of massive particles is a useful tool to model structure formation. The time of ‘radiation dominance’, on the other hand, suggests the study of ensembles of massless particles (where in the approach to the big bang singularity collisions are of course expected to play an increasing role—this is not captured by the collisionless matter model). Interestingly enough, the dynamics toward the big bang singularity in the case of massive particles is qualitatively the same as the dynamics of the massless particles model. This seems to be a generic property of Vlasov matter, see [11, 20].

The existence of a big bang (an ‘initial singularity’) is predicted by the singularity theorems of general relativity [10]. Under very general assumptions, there will exist causal curves that ‘terminate’ at a singularity, and physical quantities representing curvature or the energy density of the matter will diverge along these inextendible curves. The detailed characterization of these singularities is an important problem in general relativity and cosmology. One interesting question in this respect concerns the details of the divergence of the curvature toward the singularity (e.g., along a distinguished congruence of curves representing the (spacetime) trajectories of the matter). The well-known BKL conjecture [4] states that the approach to the singularity will in general be ‘oscillatory’, which means that appropriately rescaled (curvature) quantities will oscillate (with a rapidly increasing frequency) instead of converging monotonically toward the singularity. The paradigm of this type of behavior is the so-called ‘Mixmaster’ behavior that originates from the study of spatially homogeneous spacetimes. We refer to sections 5 and 6 of the textbook [21] for a good overview. Another important problem concerns the question of whether the asymptotic dynamics of solutions toward the singularity are sensitive to the choice of matter model or not (i.e., whether “matter matters”). Although for matter models like perfect fluids the latter seems to be the case, there are matter models such that the solutions of the Einstein-matter equations exhibit an asymptotic behavior that is different from that of vacuum solutions. The collisionless (Vlasov) matter model is a prime example, and the results of the present work are results in this vein.

To obtain a detailed characterization of the dynamics of solutions of the Einstein-Vlasov equations, it is necessary to assume a high degree of symmetry as, e.g., spatial homogeneity. The global dynamics of spatially homogeneous solutions of the Einstein-Vlasov system has been studied extensively, see, e.g., [7, 11, 16, 17, 19, 20] for applications of dynamical system theory to the problem. The reformulation of the Einstein(-Vlasov) equations as a dynamical system has proved to be highly advantageous. The reason for the success of dynamical systems methods lies in the fact that the behavior of the spacetime geometry in the neighborhood of an (initial) singularity is determined by the nature of the α -limit set of the dynamical system; see section 5.3 of [21] for a good introduction. In the simplest cases, the α -limit set is an isolated fixed point; typically, this corresponds to the spacetime curvature growing monotonically. If the α -limit set is more complicated, however, for example a heteroclinic cycle with a finite number of fixed points, then the approach to the singularity will be ‘oscillatory’. The paradigm of oscillatory behavior, the Mixmaster behavior described in the BKL conjecture, is even more intricate—it is induced by infinite heteroclinic chains.

The present work extends the results of [19] on massless Vlasov matter and [20] on massive particles. The results of these references concern the dynamics of Einstein-Vlasov solutions that satisfy the most restrictive symmetry assumptions (LRS Bianchi type I, II and III); the analysis is based on techniques from dynamical systems theory in conjunction with the use of Hubble-normalized dimensionless variables, see [21]. In the present paper we refine these techniques to investigate a class of solutions exhibiting a larger number of (true) degrees of freedom: LRS Bianchi type IX; we refer to section 2 for a definition. Our analysis employs a different set of dimensionless variables to regularize the equations and to recast the Einstein-Vlasov system into an autonomous finite-dimensional dynamical system over a compact state space. We note that the methods we use to prove our main result are closely connected with the general formalism developed in [5]. However, ensembles of massive collisionless particles do not directly fall into the class of matter models considered in [5], which makes it necessary to generalize the approach.

The paper is largely self-contained. In section 2 we discuss the collisionless matter model and give a derivation of the Einstein-Vlasov equations for the class of cosmological models we consider; the symmetry assumptions (LRS Bianchi type IX) are explained. In section 3 we reformulate the equations in terms of dimensionless variables; however, another reformulation of the equations is necessary to obtain a regular autonomous finite-dimensional dynamical system over a compact state space. The analysis of this dynamical system is performed in section 5. At the end of this section we state the main theorem: We prove that the α - and the ω -limit set of generic orbits of the dynamical system that correspond to LRS Bianchi type IX solutions of the Einstein-Vlasov equations is a heteroclinic cycle. In the concluding remarks, section 6, we illuminate the main result from a physical perspective by putting it into a broader context.

2 Derivation of the equations

Consider an ensemble of massive particles that are in geodesic motion in a ‘spacetime’ $(M, {}^4\mathbf{g})$, which is a smooth four-dimensional manifold equipped with a metric tensor field of Lorentzian signature $(-+++)$. The assumption of geodesic motion reflects the condition of absence of interactions between the particles other than gravity; since the particles interact solely through the gravitational field they create collectively, the governing equations are Einstein’s field equations of general relativity. This type of matter is commonly called ‘collisionless matter’, since, in particular, interactions by collisions are excluded. Examples of physical systems that are believed to be well approximated by the collisionless matter model in gravity are galaxies or galaxy clusters; in the former case, the particles are the stars of the galaxy, while in the cosmological setting, the particles are the galaxies of the cluster [1, 2, 3, 15].

The ensemble of particles is represented by a ‘particle distribution function’ $f_p : \mathbb{R}^+ \times UM \rightarrow [0, \infty)$, where $UM \subset TM$ is the bundle of unit mass shells (four-velocity hyperboloids), i.e., the subset of the tangent bundle TM given by the condition ${}^4\mathbf{g}(u, u) = -1$ (where u is a future-directed four-velocity). Let $x \in M$ and $u \in UM$ be a four-velocity at x ; let vol_{UM} denote the induced volume element on UM ; then

$$f_p(m, x, u) \text{vol}_{UM} dm$$

represents the proper number density of those particles whose mass is in an interval of (infinitesimal) length dm around m and whose four-velocity is in an (infinitesimal) volume vol_{UM} containing u . The proper (rest) mass density of the particles whose four-velocity is in a volume vol_{UM} containing u is

$$f(x, u) \text{vol}_{UM} = \left(\int_{\mathbb{R}^+} m f_p(m, x, u) dm \right) \text{vol}_{UM}; \quad (1)$$

this relation defines the ‘mass distribution function’ f .

Let (t, x^i) be a system of coordinates on M and $\{e_0, e_i\}$ be a frame, e.g., the coordinate frame $\{\partial_t, \partial_{x^i}\}$; we assume that e_0 is (future-directed) timelike and e_i spacelike, $i = 1, 2, 3$. Then the spatial components u^i of the four-velocity (w.r.t. the frame) are coordinates on the hyperboloid UM and we can express the invariant measure on UM as $\text{vol}_{UM} = \sqrt{|\det {}^4\mathbf{g}|} |u_0|^{-1} du^1 du^2 du^3$, where u_0 is determined from u^i by the normalization relation ${}^4\mathbf{g}(u, u) = g_{\mu\nu} u^\mu u^\nu = -1$. We adhere to the convention that spacetime indices are denoted by Greek letters, whose range is $0, 1, 2, 3$, while Latin indices are spatial indices and take the values $1, 2, 3$. We use the Einstein summation convention.

Regarding f (and f_p) as functions of the coordinates t, x^i, u^j (and m), $i, j = 1, 2, 3$, we find that the energy-momentum tensor of the ensemble is given by

$$T^{\mu\nu} = \int \text{vol}_{UM} \int dm m f_p u^\mu u^\nu = \int \text{vol}_{UM} f u^\mu u^\nu = \int f u^\mu u^\nu \sqrt{|\det {}^4\mathbf{g}|} |u_0|^{-1} du^1 du^2 du^3, \quad (2)$$

where $u_0 = u_0(u^1, u^2, u^3)$ by the condition ${}^4\mathbf{g}(u, u) = g_{\mu\nu} u^\mu u^\nu = -1$.

Remark. A common special case is the case where the ensemble of particles consists of one species of particles with equal mass \mathbf{m} , which corresponds to $f_p(m, x, u) = \delta(m - \mathbf{m}) \bar{f}(x, u)$. In this context it is customary to use the four-momentum $v = \mathbf{m}u$ as the variable of the distribution function; taking the relation between the volume elements on the unit mass shell UM and the mass shell $PM = \{v | {}^4\mathbf{g}(v, v) = -\mathbf{m}^2\}$ into account, we see that $\bar{f}(x, v) \text{vol}_{PM}$ with $\bar{f}(x, v) = \mathbf{m}^{-3} \bar{f}(x, u)$ represents the proper number density. Then $f(x, u) = \mathbf{m}^4 \bar{f}(x, v)$ and (2) becomes

$$T^{\mu\nu} = \mathbf{m}^{-4} \int f(x, \frac{v}{\mathbf{m}}) v^\mu v^\nu \sqrt{|\det {}^4\mathbf{g}|} \frac{dv^1 dv^2 dv^3}{|v_0|} = \int \bar{f}(x, v) v^\mu v^\nu \sqrt{|\det {}^4\mathbf{g}|} \frac{dv^1 dv^2 dv^3}{|v_0|},$$

where v_0 is determined from v^i by the mass shell relation ${}^4\mathbf{g}(v, v) = g_{\mu\nu} v^\mu v^\nu = -\mathbf{m}^2$.

Remark. The formalism to describe ensembles of massless particles is analogous. In the massless case, the domain of the distribution function is the bundle of future light cones, i.e., the set of future-directed null vectors. Accordingly, the energy-momentum tensor is given by (2), where u_0 is determined from u^i by the condition ${}^4\mathbf{g}(u, u) = g_{\mu\nu} u^\mu u^\nu = 0$.

Both the particle distribution function f_p and the mass distribution function f satisfy the Vlasov equation

$$\partial_t f + \frac{u^j}{u^0} \partial_{x^j} f - \frac{1}{u^0} \Gamma_{\mu\nu}^j u^\mu u^\nu \partial_{u^j} f = 0, \quad (3)$$

which reflects the condition of geodesic motion of the particles; $\Gamma_{\nu\sigma}^\mu$ are the Christoffel symbols. Note in particular that the characteristic curves of the Vlasov equation, along which f is constant, coincide with the lift on UM of the spacetime geodesics. The gravitational interaction of the particles is modeled by the Einstein-Vlasov system, i.e., by coupling (3) to the Einstein equations of general relativity,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}. \quad (4)$$

In these equations, $R_{\mu\nu}$ is the Ricci tensor of the metric ${}^4\mathbf{g}$ and $R = g^{\mu\nu} R_{\mu\nu}$ the Ricci scalar; for $T_{\mu\nu}$ we use the energy-momentum tensor (2) representing the Vlasov matter. We adopt units such that $8\pi G = c = 1$, where G is Newton's gravitational constant and c the speed of light.

In this paper we consider spatially homogeneous spacetimes of Bianchi type IX that are locally rotationally symmetric (LRS), i.e., spacetimes of the form $M = I \times S^3$ (where I is an interval of \mathbb{R}) with metric

$${}^4\mathbf{g} = -dt^2 + g_{11}(t) \hat{\omega}^1 \otimes \hat{\omega}^1 + g_{22}(t) (\hat{\omega}^2 \otimes \hat{\omega}^2 + \hat{\omega}^3 \otimes \hat{\omega}^3), \quad (5)$$

where $\{\hat{\omega}^1, \hat{\omega}^2, \hat{\omega}^3\}$ is a time-independent coframe on S^3 that satisfies $d\hat{\omega}^1 = -\hat{\omega}^2 \wedge \hat{\omega}^3$ (and cyclic permutations). As proved in [13], the general solution of the Vlasov equation (3) on a background

spacetime with metric (5) can be expressed as

$$f = f_0(u_1, (u_2)^2 + (u_3)^2), \quad (6)$$

where $u_i = g_{ij}u^j$, $i = 1, 2, 3$, and $f_0 : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an arbitrary sufficiently smooth function.

Let us compute (2) from (6). Denoting by g the spatial Riemannian metric we have $|\det {}^4\mathbf{g}| = \det g$ and we find $du^1 du^2 du^3 = (\det g)^{-1} du_1 du_2 du_3$; moreover, $|u_0|^2 = 1 + g^{11}u_1^2 + g^{22}(u_2^2 + u_3^2)$. Therefore, the energy density $\rho = T_{00}$ is given by

$$\rho = (\det g)^{-1/2} \int f_0 (1 + g^{11}u_1^2 + g^{22}(u_2^2 + u_3^2))^{1/2} du_1 du_2 du_3 \quad (7a)$$

and the principal pressures $p_1 = T_1^1$, $p_2 = T_2^2$, $p_3 = T_3^3$ are

$$p_i = (\det g)^{-1/2} \int f_0 g^{ii} u_i^2 (1 + g^{11}u_1^2 + g^{22}(u_2^2 + u_3^2))^{-1/2} du_1 du_2 du_3, \quad (7b)$$

where there is no summation over i . Since f_0 is the function (6), we find $p_2 = p_3$.

Remark. The energy density and the principal pressures (7) depend on the arbitrary function f_0 . This function can be interpreted as the ‘initial data’ for f at some time $t = t_0$, because $f(t_0, u^1, u^2, u^3) = f_0(g_{11}(t_0)u^1, (g_{22}(t_0))^2((u^2)^2 + (u^3)^2))$. Once the initial data f_0 is prescribed, ρ and p_1, p_2 are functions of the metric components, which can be interpreted as implicit relations between the principal pressures and the energy density, i.e., as an ‘equation of state’. To compare, let us briefly recall the perfect fluid matter model. In the perfect fluid case, once an equation of state $p = p(\rho)$ is prescribed, the energy density and the pressure are determined by the constraints. The evolution equations for the matter—the Euler equations—are equivalent to the conservation of the energy momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, and are thus contained in the Einstein equations (4) (through the Bianchi identities). In contrast, the Vlasov equation (3) is independent; initial data for the first order equation (3) has to be prescribed in order to obtain a solution.

For the pair (5)–(6) to be a candidate for a solution of the Einstein-Vlasov system, the energy momentum tensor must be compatible with the LRS assumption, i.e., diagonal and $T_2^2 = T_3^3$. This can be achieved by restricting to distribution functions (6) that are invariant under the transformation $u_1 \rightarrow -u_1$. These distribution functions are called *reflection symmetric*, see [16]. Besides reflection symmetry, for technical reasons, we also assume that f_0 has *split support*, which means that the support of f_0 does not intersect any of the axes.

Remark. The (rest) mass current density of particles is given as

$$N^\mu = (\det g)^{-1/2} \int f_0 u^\mu |u_0|^{-1} du_1 du_2 du_3$$

and satisfies $\nabla_\mu N^\mu = 0$, which expresses the conservation of (rest) mass. A straightforward consequence of the assumption of reflection symmetry is that $N^i = 0$, i.e., the current density is orthogonal to the hypersurfaces $t = \text{const.}$ The matter model can thus interpreted as being ‘non-tilted’.

The Einstein equations (4) split into the Hamiltonian constraint equation

$$9H^2 - ((k_1^1)^2 + 2(k_2^2)^2) + R = 2\rho, \quad (8a)$$

and the evolution equations

$$\partial_t g^{ii} = 2k^i_i g^{ii}, \quad (8b)$$

$$\partial_t k^i_i = R^i_i - 3Hk^i_i - p_i + \frac{1}{2}(p_1 + 2p_2 - \rho), \quad (8c)$$

where there is no summation over i . The quantities k^1_1 and $k^2_2 = k^3_3$ are the components of the second fundamental form of the hypersurfaces $t = \text{const}$, the Hubble scalar H is defined by $H = -\frac{1}{3} \text{tr } k$.

$$R^1_1 = \frac{1}{2} \frac{(g^{22})^2}{g^{11}}, \quad R^2_2 = R^3_3 = g^{22} - \frac{1}{2} \frac{(g^{22})^2}{g^{11}} \quad (8d)$$

are the non-zero components of the Ricci tensor, and $R = R^1_1 + 2R^2_2$ is the Ricci scalar. There is another constraint equation, the momentum constraint, that is satisfied identically by our assumptions.

It is well known that the maximal interval of existence of solutions of the Einstein-Vlasov system (7)–(8) is of the form (t_-, t_+) , where $|t_{\pm}| < \infty$; we refer to [17, 18]. After a time translation we may assume that the singularity in the past is at $t_- = 0$. The Einstein vacuum equations are recovered from (7)–(8) by setting $f_0 \equiv 0$. The asymptotic behavior of vacuum solutions is characterized by the existence of positive constants a_{\pm}, b_{\pm} , such that

$$g_{11}(t) = a_{\pm}(t - t_{\pm})^2 (1 + o(1)), \quad g_{22}(t) = g_{33}(t) = b_{\pm} + o(1) \quad (t \rightarrow t_{\pm}). \quad (9)$$

This behavior is not exclusive to vacuum solutions; on the contrary, (9) is the typical behavior of solutions associated with a large variety of matter sources, (non-stiff) perfect fluids being the prime example [5]. Since the metric

$$T: \quad -dt^2 + a(t - t_{\pm})^2 (dx^1)^2 + b((dx^2)^2 + (dx^3)^2) \quad (10a)$$

with $a, b = \text{const}$ is the so-called Taub solution (flat LRS Kasner solution), one refers to (9) as an approach to the Taub solution. (However, the spacetime associated with the metric (10a) is $(0, \infty) \times T^3$ instead of $I \times S^3$ and thus of Bianchi type I; the Taub solution does not satisfy (8d).) The second class of vacuum LRS solutions of Bianchi type I (i.e., on $(0, \infty) \times T^3$) is the class of non-flat LRS solutions

$$Q: \quad -dt^2 + a(t - t_{\pm})^{-2/3} (dx^1)^2 + b(t - t_{\pm})^{4/3} ((dx^2)^2 + (dx^3)^2); \quad (10b)$$

as opposed to (10a), the class Q does not play any particular role in the context of the asymptotic dynamics of vacuum solutions (or perfect fluid solutions) of (7)–(8).

In this paper we are interested in the asymptotic behavior of solutions of the Einstein-Vlasov system (7)–(8) as $t \rightarrow t_{\pm}$. Our main result can be informally stated as follows: *In the limit $t \rightarrow t_{\pm}$, generic solutions of the system (7)–(8) oscillate between the Taub class and the class of non-flat LRS Kasner solutions.* The asymptotic behavior of solutions of the Einstein-Vlasov system is thus qualitatively different from that of vacuum solutions. Therefore, “collisionless matter matters”, as opposed to (non-stiff) perfect fluid matter and several other matter sources which do not affect the structure of the singularity.

Remark. The fact that the behavior toward the singularity of cosmological models with collisionless matter is qualitatively different from that of vacuum and perfect fluid models is known from LRS Bianchi type II models [19, 20]. The occurrence of oscillations in the asymptotic dynamics of LRS models that are induced by the anisotropy of the matter model has subsequently been studied in some detail, e.g., in [5, 9].

3 Reformulated Einstein-Vlasov equations

Let

$$n = \frac{1}{\sqrt{\det g}} = \sqrt{g^{11}(g^{22})^2} \quad \text{and} \quad s = \frac{g^{22}}{g^{11} + 2g^{22}}.$$

Note that due to (1) and (6), $n = n(t) \in (0, \infty)$ is proportional to the mass (or particle) density of the ensemble of particles (as measured w.r.t. the frame associated with (5)). The variable s , on the other hand, satisfies $s = s(t) \in (0, \frac{1}{2})$ and is a (non-linear) measure of the deviation of the metric from isotropy; $s = \frac{1}{3}$ ($\Leftrightarrow g_{11} = g_{22}$) corresponds to an isotropic metric. We further define

$$\ell = \frac{1}{1 + n^{2/3}}.$$

Since $n^{2/3} = (\det g)^{-1/3}$, $n^{2/3}$ corresponds to a length scale of the metric, and ℓ is a non-linear measure of such a scale; $\ell = 0$ corresponds to a singularity ($\det g = 0$); $\ell = 1$ to a state of infinite volume ($\det g = +\infty$). The equation satisfied by ℓ follows directly from (8b),

$$\partial_t \ell = 2H\ell(1 - \ell); \quad (11)$$

recall that $H = -\frac{1}{3} \text{tr } k$ is the Hubble scalar; accordingly, $H > 0$ means expansion, $H < 0$ contraction. For s we find $\partial_t s = -2s(1 - 2s)(k_1^1 - k_2^2)$, where $k_1^1 - k_2^2$ can be identified with (three times) the 2-2-component of the shear tensor.

We are able to express the energy density (7a) and the principal pressures (7b) in terms of ℓ and s ; defining

$$w_i = \frac{p_i}{\rho}, \quad w = \frac{p}{\rho} = \frac{1}{3} \frac{p_1 + 2p_2}{\rho} = \frac{1}{3} (w_1 + 2w_2) \quad (12)$$

we obtain

$$w_1 = (1 - \ell)(1 - 2s) \frac{\int f_0 u_1^2 \left[\ell(s^2(1 - 2s))^{1/3} + (1 - \ell)((1 - 2s)u_1^2 + s(u_2^2 + u_3^2)) \right]^{-1/2} du}{\int f_0 \left[\ell(s^2(1 - 2s))^{1/3} + (1 - \ell)((1 - 2s)u_1^2 + s(u_2^2 + u_3^2)) \right]^{1/2} du}, \quad (13a)$$

$$w_2 = (1 - \ell)s \frac{\int f_0 u_2^2 \left[\ell(s^2(1 - 2s))^{1/3} + (1 - \ell)((1 - 2s)u_1^2 + s(u_2^2 + u_3^2)) \right]^{-1/2} du}{\int f_0 \left[\ell(s^2(1 - 2s))^{1/3} + (1 - \ell)((1 - 2s)u_1^2 + s(u_2^2 + u_3^2)) \right]^{1/2} du}, \quad (13b)$$

where du abbreviates $du_1 du_2 du_3$. The assumption of split support assures that w_1, w_2 are smooth functions, since the denominator in (13) is strictly positive for all values of ℓ and s .

For our analysis it is necessary to recast the Einstein equations (8) into a different form. We use the dominant variable

$$D = \sqrt{H^2 + \frac{1}{3} g^{22}} = \sqrt{\frac{1}{9} (\text{tr } k)^2 + \frac{1}{3} g^{22}} \quad (14a)$$

to define normalized dimensionless variables according to

$$H_D = \frac{H}{D}, \quad \Sigma_+ = \frac{k_1^1 - k_2^2}{3D}, \quad M_1 = \frac{1}{D} \frac{g^{22}}{\sqrt{g^{11}}}, \quad \Omega = \frac{\rho}{3D^2}. \quad (14b)$$

In addition we replace the cosmological time t by a rescaled time variable τ via

$$\frac{d}{d\tau} = \left(\quad \right)' = \frac{1}{D} \frac{d}{dt}. \quad (14c)$$

Rewriting the Einstein equations (8) in the new variables, we obtain a decoupled ODE for D and

a system of ODEs that we call the *reduced dynamical system*:

$$H'_D = -(1 - H_D^2)(q - H_D \Sigma_+) , \quad (15a)$$

$$\Sigma'_+ = -(2 - q)H_D \Sigma_+ - (1 - H_D^2)(1 - \Sigma_+^2) + \frac{1}{3} M_1^2 + \Omega (w_2(\ell, s) - w_1(\ell, s)) , \quad (15b)$$

$$M'_1 = M_1 (q H_D - 4 \Sigma_+ + (1 - H_D^2) \Sigma_+) , \quad (15c)$$

$$\ell' = 2 H_D \ell (1 - \ell) . \quad (15d)$$

In the system (15), q is the so-called deceleration parameter, $q = 2\Sigma_+^2 + \frac{1}{2}(1 + 3w)\Omega$; in addition, Ω is determined from the variables Σ_+ and M_1 by the Hamiltonian constraint (8a), and (14) is used to express s as a function of H_D and M_1 , i.e.,

$$\Omega = 1 - \Sigma_+^2 - \frac{1}{12} M_1^2 , \quad s = \left(2 + \frac{3(1 - H_D^2)}{M_1^2} \right)^{-1} . \quad (16)$$

The system (15) is a closed system that completely describes the dynamics of LRS Bianchi type IX Einstein-Vlasov models. Note that the r.h.s. of (15) contains the functions (13) that are determined by an integration over $f_0 = f_0(u_1, u_2, u_3)$. This means that, in particular, the r.h.s. of the dynamical system (15) depends on the initial data f_0 , which is an interesting feature of the problem. A more detailed derivation of (15) is given in [5].

Remark. The dynamical system (15) is invariant under the discrete symmetry

$$\tau \rightarrow -\tau , \quad H_D \rightarrow -H_D , \quad \Sigma_+ \rightarrow -\Sigma_+ .$$

Hence the qualitative behavior of solutions in the limit $\tau \rightarrow +\infty$ mirrors the behavior at $\tau \rightarrow -\infty$, and we may thus restrict ourself to study the latter.

The state space for the dynamical system (15) is given by

$$\mathcal{E}_{\text{IX}} = \mathcal{X}_{\text{IX}} \times (0, 1) , \quad \text{where } \mathcal{X}_{\text{IX}} = \left\{ (H_D, \Sigma_+, M_1) \mid H_D \in (-1, 1) , M_1 > 0 , \Sigma_+^2 + \frac{1}{12} M_1^2 < 1 \right\} ,$$

see Fig. 1. The set \mathcal{E}_{IX} is relatively compact; the system (15) is smooth on \mathcal{E}_{IX} . However, (15) does not admit a regular extension to the entire boundary $\partial \mathcal{E}_{\text{IX}}$, which is because the variable s does not have a well-defined limit when $M_1 \rightarrow 0$ and $H_D^2 \rightarrow 1$ simultaneously. This defect will be remedied by introducing the equivalent system (19).

Remark. Let us briefly comment on the equations describing the dynamics of cosmological models with different matter sources. The reduced dynamical system for perfect fluid matter is obtained from (15) by formally setting $w_1 = w_2 = w = \text{const}$ in (15b), which reflects the isotropy of the matter model (and the assumption of a linear equation of state). The equation for ℓ decouples from the remaining equations and the reduced dynamical system becomes the set of equations (15a)–(15c) on the state space \mathcal{X}_{IX} . The reduced dynamical system in the case of an ensemble of massless particles is characterized by a decoupling of the equation for ℓ as well. This is because, in the massless case, the renormalized principal pressures are obtained from (13) by formally setting $\ell = 0$. The reduced dynamical system thus becomes the set of equations (15a)–(15c). We therefore find that \mathcal{X}_{IX} is the state space for both the perfect fluid case and the massless Vlasov case. We remark that this is not so for the lower Bianchi types. As shown in [19], the state space for massless Vlasov particles has one dimension more than the state space for perfect fluids when the Bianchi type is I, II or III. In the Bianchi type IX case, the state spaces are identical, but another difference occurs: While the reduced dynamical system for perfect fluids admits a smooth extension to the boundary of \mathcal{X}_{IX} , the Vlasov case is defective in this respect. Loosely speaking, the dynamics of Vlasov matter for massless particles does not live naturally in the state space \mathcal{X}_{IX} .

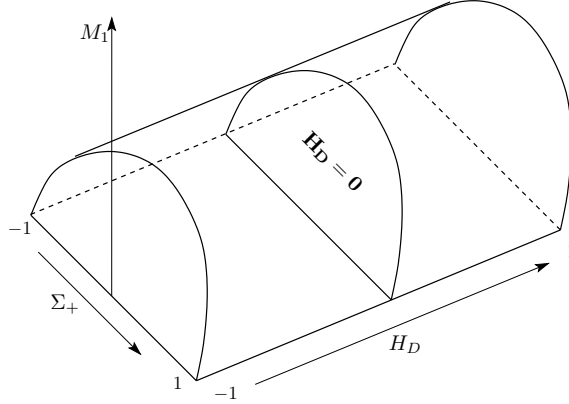


Figure 1: The set \mathcal{X}_{IX} .

4 Basic lemmas

We use the system (15) to prove two basic lemmas.

Lemma 1. *For every Bianchi type IX solution with Vlasov matter there exists $\tau_0 \in \mathbb{R}$ such that*

- $H_D(\tau) > 0 \ \forall \tau < \tau_0$ and $H_D(\tau)$ is bounded away from zero as $\tau \rightarrow -\infty$;
- $H_D(\tau_0) = 0$;
- $H_D(\tau) < 0 \ \forall \tau > \tau_0$ and $H_D(\tau)$ is bounded away from zero as $\tau \rightarrow \infty$.

Proof. The average pressure p of collisionless matter is non-negative; eqs. (12) and (13) imply $w = p/\rho \geq 0$. Therefore the general result by Lin and Wald [12] applies: Every Bianchi type IX model (with collisionless matter) possesses an initial singularity (‘big bang’), expands initially, then reaches a time when the spatial volume is maximal, and finally recontracts to terminate in a singularity (‘big crunch’). Hence there exists τ_0 such that $H_D(\tau) > 0 \ \forall \tau < \tau_0$, $H_D(\tau_0) = 0$, and $H_D(\tau) < 0 \ \forall \tau > \tau_0$. It remains to prove that there exists a positive ϵ such that $H_D(\tau) \geq \epsilon$ ($H_D(\tau) \leq -\epsilon$) for all sufficiently small (large) τ . To do so assume the contrary, i.e., the existence of a solution whose α -limit set has a non-empty intersection with the plane $H_D = 0$. We first note that

$$H'_D|_{H_D=0} = -2\Sigma_+^2 - \frac{1}{2}(1+3w)\Omega,$$

which is negative unless $\Omega = 0$ and $\Sigma_+ = 0$. Second,

$$H'_D|_{H_D=0, \Omega=0, \Sigma_+=0} = 0, \quad H''_D|_{H_D=0, \Omega=0, \Sigma_+=0} = 0, \quad H'''_D|_{H_D=0, \Omega=0, \Sigma_+=0} = -36 < 0.$$

Let P be an α -limit point of the solution such that P lies on $H_D = 0$. Together with P , the entire orbit through P is contained in the α -limit set; hence there exists a sequence of times $(\tau_n)_{n \in \mathbb{N}}$ with $\tau_n \rightarrow -\infty$ ($n \rightarrow \infty$) such that $H_D(\tau_n - \delta) > 0$, $H_D(\tau_n) = 0$, $H_D(\tau_n + \delta) < 0$ for a sufficiently small δ ; a contradiction. \square

Remark. A posteriori, by Lemma 1, we find $H_D(\tau) \rightarrow \pm 1$ as $\tau \rightarrow \mp \infty$. A note of caution: There exist anisotropic matter models other than collisionless matter such Lemma 1 is false; for such matter models there exist (typical) solutions that do not recollapse but expand forever (i.e., $H_D(\tau) > 0 \ \forall \tau$); we refer to [8].

Lemma 2. *Let P be an α -limit point of a Bianchi type IX solution with Vlasov matter as represented by an orbit of (15). Then*

$$\ell|_P = 0.$$

Proof. The result is a simple consequence of eq. (15d) for ℓ and Lemma 1. \square

Remark. Lemma 2 has a straightforward physical interpretation. Since (15) with $\ell = 0$ describes the dynamics of solutions of the Einstein-Vlasov equations with massless particles, see the remark at the end of section 3, we find that the asymptotic dynamics toward the past (and future) singularity in the massive Vlasov case are governed by the the dynamics of the massless Vlasov case.

5 Analysis and main result

By Lemma 1, the subset $H_D > 0$ of the state space \mathcal{E}_{IX} is past-invariant under the flow of (15); this makes it possible to transform (15) to a different system of equations that is equivalent to (15) on the subset $H_D > 0$.

Let

$$M_1^2 = 3r \sin \vartheta, \quad (17a)$$

$$(1 - H_D^2) = 2r \cos \vartheta, \quad (17b)$$

$$\Sigma_+ = \text{unchanged}, \quad (17c)$$

where we assume that $(H_D, M_1, \Sigma_+) \in \mathcal{X}_{\text{IX}}^+ = \mathcal{X}_{\text{IX}} \cap \{H_D > 0\}$. The transformation of variables from (H_D, M_1, Σ_+) to (r, ϑ, Σ_+) , where $r > 0$ and $0 < \vartheta < \frac{\pi}{2}$ is required, is a diffeomorphism on the domain $(H_D, M_1, \Sigma_+) \in \mathcal{X}_{\text{IX}}^+$. We define the domain $\mathcal{Y}_{\text{IX}}^+$ of the variables (r, ϑ, Σ_+) to be the preimage of that domain under the transformation (17). We obtain $r \cos \vartheta < \frac{1}{2}$ from (17b); the constraint $1 - \Sigma_+^2 - \frac{1}{12} M_1^2 = \Omega > 0$ implies $r \sin \vartheta < 4(1 - \Sigma_+^2)$. Therefore, $\mathcal{Y}_{\text{IX}}^+$ can be written as

$$\mathcal{Y}_{\text{IX}}^+ = \left\{ r > 0, \vartheta \in (0, \frac{\pi}{2}), \Sigma_+ \in (-1, 1) \mid r < \min \left[\frac{1}{2 \cos \vartheta}, \frac{4(1 - \Sigma_+^2)}{\sin \vartheta} \right] \right\}, \quad (18)$$

see Fig. 2(a).

In the new variables, the dynamical system (15) takes the form

$$r' = 2r (H_D(q - H_D \Sigma_+) - 3\Sigma_+ \sin^2 \vartheta), \quad (19a)$$

$$\vartheta' = -3\Sigma_+ \sin(2\vartheta), \quad (19b)$$

$$\Sigma_+' = r \sin \vartheta - 1 + (H_D - \Sigma_+)^2 + H_D \Sigma_+ (q - H_D \Sigma_+) + \Omega (w_2(\ell, s) - w_1(\ell, s)), \quad (19c)$$

$$\ell' = 2H_D \ell (1 - \ell). \quad (19d)$$

where $q = 2\Sigma_+^2 + \frac{1}{2}(1 + 3w)\Omega$ and $\Omega = 1 - \Sigma_+^2 - \frac{1}{4}r \sin \vartheta$. In addition, H_D and s are regarded as functions of r and ϑ in (19),

$$H_D = \sqrt{1 - 2r \cos \vartheta}, \quad s = \frac{1}{2} \frac{\tan \vartheta}{1 + \tan \vartheta}. \quad (19e)$$

Note that these are smooth functions of r and ϑ on the state space $\mathcal{Y}_{\text{IX}}^+$. In particular,

$$\frac{\partial s}{\partial \vartheta} = \frac{1}{\sin(2\vartheta)} 2s(1 - 2s) = \frac{1}{2} \frac{1}{1 + \sin(2\vartheta)}, \quad \text{hence} \quad \frac{1}{4} \leq \frac{\partial s}{\partial \vartheta} \leq \frac{1}{2} \quad \forall \vartheta \in [0, \frac{\pi}{2}]$$

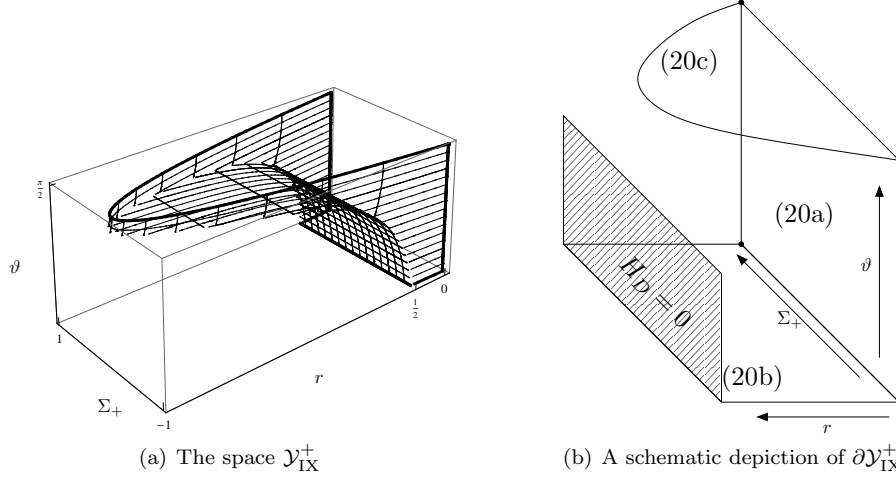


Figure 2: The space $\mathcal{Y}_{\text{IX}}^+$ and its boundary. The various components of the boundary are defined in eqs. (20).

and $\partial s / \partial \vartheta = \frac{1}{2}$ at $\vartheta = 0$ and $\vartheta = \frac{\pi}{2}$.

The flow of the system (19) on the state space $\mathcal{Y}_{\text{IX}}^+ \times (0, 1)$ is well defined in the past direction of time. This reflects the past invariance of the original domain $\mathcal{X}_{\text{IX}}^+ \times (0, 1)$. Accordingly, an orbit of (19) represents the expanding phase of an LRS Bianchi type IX solution with Vlasov matter; conversely, the expanding phase of every LRS type IX model is represented by an orbit of (19).

In contrast to the system (15), the dynamical system (19) on the state space $\mathcal{Y}_{\text{IX}}^+ \times (0, 1)$ *admits a regular extension to the boundaries*. By the boundaries of (18) we mean

$$r = 0 : \quad \left\{ \vartheta \in \left(0, \frac{\pi}{2}\right), \Sigma_+ \in (-1, 1) \right\}, \quad (20a)$$

$$\vartheta = 0 : \quad \left\{ r \in \left(0, \frac{1}{2}\right), \Sigma_+ \in (-1, 1) \right\}, \quad (20b)$$

$$\vartheta = \frac{\pi}{2} : \quad \left\{ r \in (0, 4(1 - \Sigma_+^2)), \Sigma_+ \in (-1, 1) \right\}, \quad (20c)$$

$$\Omega = 0 : \quad \left\{ r = \min \left[\frac{1}{2 \cos \vartheta}, \frac{4(1 - \Sigma_+^2)}{\sin \vartheta} \right] = \frac{4(1 - \Sigma_+^2)}{\sin \vartheta}, \vartheta \in \left(0, \frac{\pi}{2}\right), \Sigma_+ \in (-1, 1) \right\}, \quad (20d)$$

and the closures of these invariant sets. The set

$$\left\{ r = \min \left[\frac{1}{2 \cos \vartheta}, \frac{4(1 - \Sigma_+^2)}{\sin \vartheta} \right] = \frac{1}{2 \cos \vartheta}, \vartheta \in \left(0, \frac{\pi}{2}\right), \Sigma_+ \in (-1, 1) \right\}$$

is the preimage of the plane $H_D = 0$ under (17); it is not an invariant set; by Lemma 1 it is irrelevant for our considerations. This exhausts the list of boundaries of (18).

Lemma 3. *Let P be an α -limit point of an orbit of (19) (i.e., of a Bianchi type IX solution with Vlasov matter). Then*

$$r|_P = 0 \quad \text{or} \quad \vartheta|_P = \frac{\pi}{2}.$$

Proof. We consider the function

$$Y = H_D^6 \frac{\tan \vartheta}{r^3 \cos^3 \vartheta},$$

which is smooth and positive on $\mathcal{Y}_{\text{IX}}^+$ and on the boundary subset $\Omega = 0$, see (20d). We find that

$$Y' = -12Y(\Sigma_+^2 + \Omega), \quad Y'''|_{\Sigma_+=0, \Omega=0} = -24Y(3 + H_D^2), \quad (21)$$

hence Y is strictly monotonically decreasing along every orbit. This excludes the existence of α -limit points in $\mathcal{Y}_{\text{IX}}^+$ and on the boundary $\Omega = 0$. Suppose that there exists an orbit that possesses an α -limit point P with $\vartheta|_P = 0$ (and $r|_P > 0$). Then there exists a sequence of times $(\tau_n)_{n \in \mathbb{N}}$ with $\tau_n \rightarrow -\infty$ ($n \rightarrow \infty$) such that $Y(\tau_n) \rightarrow 0$ ($n \rightarrow \infty$) along the orbit; a contradiction to (21). \square

Lemma 3 suggests to study the dynamical system that is induced by (19) on the boundaries $r = 0$ and $\vartheta = \frac{\pi}{2}$, see (20a) and (20c).

The boundary subset $r = 0$. The dynamical system (19) induces the system

$$\vartheta' = -3\Sigma_+ \sin(2\vartheta), \quad (22a)$$

$$\Sigma_+' = -(1 - \Sigma_+^2)[\Sigma_+ - (w_1(0, s) - w_2(0, s))], \quad (22b)$$

on the boundary subset represented by $r = 0$ and $\ell = 0$, see (20a). From (13) we see that

$$w_1(0, s) = (1 - 2s) \frac{\int f_0 u_1^2 [(1 - 2s)u_1^2 + s(u_2^2 + u_3^2)]^{-1/2} du}{\int f_0 [(1 - 2s)u_1^2 + s(u_2^2 + u_3^2)]^{1/2} du}, \quad (23a)$$

$$w_2(0, s) = s \frac{\int f_0 u_2^2 [(1 - 2s)u_1^2 + s(u_2^2 + u_3^2)]^{-1/2} du}{\int f_0 [(1 - 2s)u_1^2 + s(u_2^2 + u_3^2)]^{1/2} du}, \quad (23b)$$

which implies that $w = \frac{1}{3}(w_1 + 2w_2) = \frac{1}{3}$. Recall that in the context of (22), $s = s(\vartheta)$, see (19e).

It is straightforward to see that $w_1(0, s) - w_2(0, s) = 1 - 3w_2(0, s)$ is a strictly monotonic function [19]. Since $w_2(0, 0) = 0$ and $w_2(0, \frac{1}{2}) = \frac{1}{2}$, there is a unique value $s_0 \in (0, \frac{1}{2})$ such that $w_2(0, s_0) = \frac{1}{3}$. Let ϑ_0 denote the (unique) value of ϑ associated with s_0 by (19e). We conclude that there exists a unique fixed point F in the interior of the set $r = 0$, $\ell = 0$:

F The fixed point F is given by $r = 0$, $\vartheta = \vartheta_0$, $\Sigma_+ = 0$, $\ell = 0$.

The remaining fixed points of the system (22) are located on the boundary of the subset $r = 0$; these are given by $\vartheta = 0$, $\vartheta = \frac{\pi}{2}$, and $\Omega = 0$ (with $\Sigma_+ = \pm 1$).

T_# The *Taub point* $T_{\#}$ is given by $r = 0$, $\vartheta = \frac{\pi}{2}$, $\Sigma_+ = -1$, $\ell = 0$.

Q_# The *non-flat LRS point* $Q_{\#}$ is given by $r = 0$, $\vartheta = \frac{\pi}{2}$, $\Sigma_+ = 1$, $\ell = 0$.

R_# The point $R_{\#}$ is given by $r = 0$, $\vartheta = \frac{\pi}{2}$, $\Sigma_+ = \frac{1}{2}$, $\ell = 0$.

T_b The *Taub point* T_b is given by $r = 0$, $\vartheta = 0$, $\Sigma_+ = -1$, $\ell = 0$.

Q_b The *non-flat LRS point* Q_b is given by $r = 0$, $\vartheta = 0$, $\Sigma_+ = 1$, $\ell = 0$.

The physical *interpretation* of the fixed points is straightforward. At the fixed point F , since $w_1 = w_2 = w_3 = w = \frac{1}{3}$, the principal pressures are identical and the matter is thus isotropic. Since $\Sigma_+ = 0$, we have $k_1^1 = k_2^2 = k_3^3$, i.e., the geometry is isotropic as well. Accordingly, the fixed point F represents the Friedmann-Lemaître-Robertson-Walker model,

$$g_{11} = g_{22} = g_{33} = at^2 \quad (a > 0), \quad (24)$$

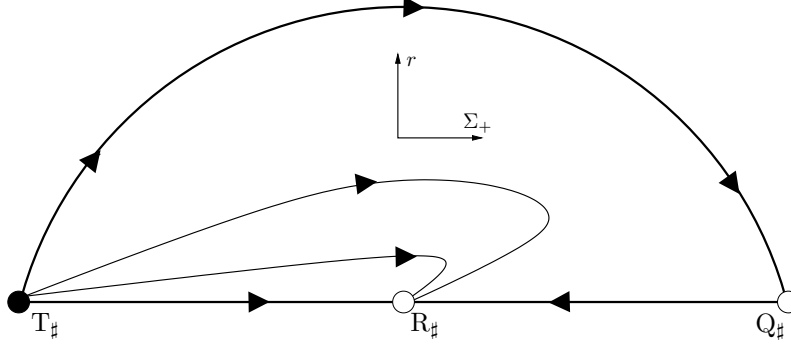


Figure 4: The flow induced by (19) on the two-dimensional boundary subset determined by $\vartheta = \frac{\pi}{2}$ and $\ell = 0$.

on the boundary subset represented by $\vartheta = \frac{\pi}{2}$ and $\ell = 0$, see (20c). Here we have used that $s = \frac{1}{2}$ when $\vartheta = \frac{\pi}{2}$, see (19e), which implies that $w_1(0, s) = 0$ and $w_2(0, s) = \frac{1}{2}$, see (23).

The one-dimensional boundary of the system (25) consists of a part where $\Omega = 0$ and a part $r = 0$; on the latter, the systems (22) and (25) intersect.

Lemma 5. *The system (25) on the closure of the subset $\vartheta = \frac{\pi}{2}$ (and $\ell = 0$) gives rise to the flow depicted in Fig. 4.*

Proof. The absence of fixed points, periodic orbits and heteroclinic cycles in the interior of this set implies, by the Poincaré-Bendixson theorem [14], that the α - and ω -limit points of orbits must be located on the boundary. A local dynamical systems analysis then yields the claim of the lemma. \square

This completes our analysis of the boundary subsets. Let us return to the study of (19).

Lemma 6. *There exists a one-parameter family of orbits of (19) whose α -limit set is the fixed point F.*

Proof. A simple calculation shows that the unstable manifold of F is two-dimensional. (See Lemma 4 for the stable manifold.) \square

Lemma 7. *Assume that a point P of the (interior of the) boundary subset $r = 0$, $\ell = 0$, see (20a), is an α -limit point of an orbit of (19). Then $P = F$.*

Proof. Let the orbit under consideration be denoted by γ . Assume that $P \neq F$. Together with P, the entire orbit through P and its ω -limit point F must be in the α -limit set of γ ; see Lemma 4. Since F is a saddle point, see Lemmas 4 and 6, there is a point \hat{P} on the unstable manifold of F that is contained in $\alpha(\gamma)$. (Lemma 2 enforces \hat{P} to be located on $\ell = 0$.) Let $(\tau_n)_{n \in \mathbb{N}}$ with $\tau_n \rightarrow -\infty$ ($n \rightarrow \infty$) denote a sequence of times such that $\gamma(\tau_n) \rightarrow \hat{P}$ ($n \rightarrow \infty$); by assumption, none of the points $\gamma(\tau_n)$ are contained on the unstable manifold of F. The orbit through \hat{P} represents a solution of the Einstein-Vlasov equations that satisfies Lemma 1. By continuous dependence on initial data we can thus construct a sequence of times $\hat{\tau}_n$ with $\hat{\tau}_n \rightarrow -\infty$ ($n \rightarrow \infty$) such that $H_D(\hat{\tau}_n) \rightarrow 0$ ($n \rightarrow \infty$); this is in contradiction to Lemma 1. \square

Lemma 8. *There does not exist any point in the (interior of the) boundary subset $\vartheta = \frac{\pi}{2}$, $\ell = 0$, see (20c), that is an α -limit point of an orbit of (19).*

Proof. Assume that there exists an orbit γ such that $P \in \alpha(\gamma)$, where P lies on the (interior of the) boundary subset $\vartheta = \frac{\pi}{2}$, see (20c). Together with P , the entire orbit through P and its ω -limit point R_{\sharp} must be in the α -limit set of γ ; see Lemma 5. The fixed point R_{\sharp} is a saddle point, see Lemmas 4 and 5; hence there is a point \hat{P} on the unstable manifold of R_{\sharp} that is contained in $\alpha(\gamma)$. The unstable manifold of R_{\sharp} is two-dimensional; a particular orbit contained in it is the orbit $R_{\sharp} \rightarrow F$ depicted in Fig. 3; the orthogonal unstable direction is the ℓ -direction. Lemma 2 implies that $\ell|_{\hat{P}} = 0$, hence \hat{P} is located on the orbit $R_{\sharp} \rightarrow F$ of Fig. 3. However, this is a contradiction to Lemma 7. \square

Lemma 9. *There does not exist any point on the (interior of the) line $T_{\sharp} \rightarrow R_{\sharp} \leftarrow Q_{\sharp}$, see Figs. 3 and 4, that is an α -limit point of an orbit of (19).*

Proof. The proof is analogous to the proof of Lemma 8. \square

Summarizing the statements of the previous lemmas we obtain the main theorem of this work. We give a formal statement of the theorem using the developed framework; the less formal interpretation of the theorem is presented in the concluding remarks.

Theorem 1. *Consider an orbit of (19) representing (the expanding phase of) a Bianchi type IX solution with Vlasov matter. Then the α -limit set of this orbit is either the fixed point F , which is the non-generic case, or it is the heteroclinic cycle of orbits connecting the four fixed points T_{\flat} , T_{\sharp} , Q_{\sharp} , Q_{\flat} , see Fig. 5; this is the generic case.*

Proof. By Lemma 6 there exists a non-generic family of orbits whose α -limit set is F . Consider an orbit that is not a member of this family. Lemma 2 implies that the α -limit set must be located on the boundary $\ell = 0$ of the state space of (19). Lemma 3 enforces α -limit points to lie on the closures of the boundary subsets $r = 0$ and $\vartheta = \frac{\pi}{2}$. However, Lemmas 7 and 8 imply that points in the interior of these subsets are excluded. Finally, by Lemma 9 we find that the set that remains as the possible location of α -limit points is a one-dimensional set, the heteroclinic cycle connecting the fixed points T_{\flat} , T_{\sharp} , Q_{\sharp} , Q_{\flat} . \square

6 Concluding remarks

Let us give an interpretation of the main result, Theorem 1, in informal terms: Every LRS Bianchi type IX solution with collisionless matter (i.e., solution of the Einstein-Vlasov system) possesses an initial singularity (which is chosen to be at $t = 0$). The behavior as $t \rightarrow 0$ of generic solutions, where by ‘generic solutions’ we mean a family of solutions that corresponds to a set of initial data of full measure, is characterized by oscillations between the Taub solution (10a) and the non-flat LRS solution (10b): There exists an infinite sequence of (increasingly small) time intervals such that the components of the metric (5) are well approximated by (10a), and a sequence of time intervals where (10b) yields a good approximation; the accuracy of the approximation increases as $t \rightarrow 0$. During the transitions between (10a) and (10b) the metric takes an entirely different form; we merely note that each transition from the non-flat LRS solution to the Taub solution (which correspond to an orbit of (19) closely following the orbit $Q_{\flat} \rightarrow T_{\flat}$ in Fig. 5) is characterized by the fact that the variable Ω attains values close to 1. This means that the influence of the matter on the (asymptotic) dynamics cannot be neglected. The oscillatory behavior toward the singularity of generic LRS Bianchi type IX cosmological models with collisionless matter is thus qualitatively different from that of perfect fluid models with the same symmetry. In fact, in the LRS case, perfect fluid cosmologies of Bianchi type IX are asymptotic to the Taub solution (10a), which is also the behavior of LRS Bianchi type IX *vacuum* cosmological models. Therefore, as the ‘structure’ of

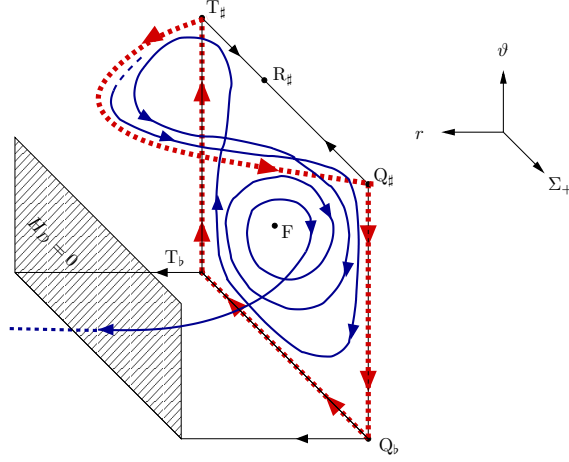


Figure 5: The projection of a generic orbit (blue line) of the dynamical system (19) to the set $\ell = 0$. Dashed red lines are orbits that lie on the boundary of the state space and connect to form a heteroclinic cycle. There exist non-generic orbits that are asymptotic to the fixed point F (not depicted).

the singularity is concerned, the assumption of perfect fluid matter does not change the vacuum behavior (‘matter does not matter’), while ‘collisionless matter matters’; Vlasov matter has an important effect on the structure of the singularity.

The occurrence of oscillatory dynamics toward the singularity is intimately connected with the (in)stability of the Kasner fixed points (i.e., the Taub points and the non-flat LRS points in the context of the present work). While stable in the context of the simplest cosmological models, which are characterized by a small number of (true) degrees of freedom and by the isotropy of the matter model, the Kasner points run the risk of losing their stability when degrees of freedom are added to the problem or when the matter model becomes anisotropic. Loss of stability is then a possible cause for the existence of heteroclinic structures which in turn induce oscillatory behavior of cosmological models. The prime example is the transition from (non-LRS) Bianchi type VI_0 and VII_0 models to (non-LRS) type VIII and IX models in the vacuum case. The dimension of the state space increases, the Kasner points lose their stability properties, and the dynamics of (vacuum) cosmological models become the oscillatory “Mixmaster” dynamics. Similarly in spirit, in the context of LRS Bianchi models with Vlasov matter, the dimension of the state space increases when one goes over from LRS type I to LRS type II models. The loss of stability induces oscillatory behavior of LRS Bianchi type II solutions with Vlasov matter, see [19, 20]. However, already a change of matter model alone (where the number of degrees of freedom of the problem are unaffected) can change the stability properties of the Kasner points: For instance, while the asymptotic dynamics of LRS Bianchi type I solution with Vlasov matter are “monotone”, one observes oscillatory behavior for LRS Bianchi type I solutions with elastic matter [9]. The present work provides another example: The state spaces and reduced dynamical systems that represent the dynamics of LRS type IX perfect fluid models and LRS type IX models with massless Vlasov matter are of the same dimension; the same is true for the state space of LRS type IX perfect fluid models with non-linear equations of state and the state space of type IX models with massive Vlasov matter considered in this paper. Despite this fact, the qualitative dynamics are radically different, which is ultimately because the Kasner points exhibit different stability properties. In brief: “Matter matters”.

Note that by the symmetry of the problem, the behavior of generic solutions toward the final

singularity (see Lemma 1) mirrors the behavior toward the initial singularity. Furthermore, we note there exists a non-generic family of LRS Bianchi type IX solutions with collisionless matter that isotropize toward the initial singularity, see Lemma 6. These solutions are asymptotic to the isotropic Friedmann-Lemaître-Robertson-Walker model (24), i.e., $g_{ij}(t) \sim t\delta_{ij}$ as $t \rightarrow 0$. Although these non-generic solutions do not influence the asymptotic behavior of generic ones, they may affect the intermediate behavior of generic solutions and thus play a relevant role in physics. We refer to [21] for a discussion on this interesting topic.

In this paper we have shown that collisionless matter has an important effect on the structure of the singularity already in the context of solutions of the Einstein equations with high symmetries (LRS Bianchi type IX). It is to be expected that this type of effect becomes even more pronounced when the degree of symmetry is reduced. In particular, an exciting question is in which manner collisionless matter will influence the so-called Mixmaster behavior of generic (non-LRS) vacuum Bianchi type IX models. Naïvely, one might expect that the two types of oscillations, the Mixmaster oscillations (which are induced, in a manner of speaking, by gravity itself) and the oscillations caused by the anisotropy of the matter model ‘intertwine’ to yield a intricate oscillatory structure. Unfortunately, it is difficult to give a well formulated conjecture, the reason being that it is not known at present how to extend the dynamical systems method to the full (non-LRS) Bianchi type IX case.

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